Asymptotic Mean Value Formulas for Solutions of General Second-Order Elliptic Equations.

Julio D. Rossi

U. Buenos Aires (Argentina), jrossi@dm.uba.ar

To the memory of Ireneo Peral.

P. Blanc (Buenos Aires),
F. Charro (Detroit),
J. J. Manfredi (Pittsburgh),

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Classical Mean Value Property

Mean value formulas characterize harmonic functions:

$$\Delta u(x) = 0 \quad \text{in } \Omega \qquad \Longleftrightarrow \qquad u(x) = \fint_{B_r(x)} u(y)\,dy \qquad \forall B_r(x) \subset \Omega.$$

(Recall Polidoro's talk)

An asymptotic statement holds:

$$\Delta u(x) = f(x) \quad \text{in } \Omega$$

$$\Longleftrightarrow$$

$$u(x) = \fint_{B_\varepsilon(x)} u(y)\,dy - \frac{\varepsilon^2}{2(n+2)} f(x) + o(\varepsilon^2) \qquad \text{as } \varepsilon \to 0.$$

Operators involving bounded sets of coefficients

First, we consider differential operators of the form

$$F(x, D^2u(x)) = \inf_{A\in\mathcal{A}_x} \operatorname{trace}(A^t D^2u(x)A).$$

Here, $\mathcal{A}_x$ is a bounded subset of $S^n_+(\mathbb{R})$.

One can also consider convex operators of the form

$$F(x, D^2u(x)) = \sup_{A\in\mathcal{A}_x} \operatorname{trace}(A^t D^2u(x)A).$$

Theorem

A function $u \in C(\Omega)$ is a viscosity solution to

$$F(x, D^2u(x)) = \inf_{A \in \mathcal{A}_x} \operatorname{trace}(A^t D^2 u(x) A) = f(x) \quad \text{in } \Omega,$$

if and only if

$$u(x) = \inf_{A \in \mathcal{A}_x} \fint_{B_\varepsilon(0)} u(x+Ay)\, dy - \frac{\varepsilon^2}{2(n+2)} f(x) + o(\varepsilon^2), \qquad \text{as } \varepsilon \to 0$$

in the viscosity sense.
(subsolution, $\leq$; supersolution, $\geq$)

Remark: $z = x + Ay \in x + AB_\varepsilon(0)$, then $|x - z| \leq C\varepsilon$ (the mean value formula is local).

Examples

We will denote the eigenvalues of a matrix $M \in S^n(\mathbb{R})$ by $\lambda_1(M) \le \lambda_2(M) \le \cdots \le \lambda_n(M)$.

- Pucci operators

$$\mathcal{M}^-_{\theta,\Theta}(D^2u) = \theta \sum_{\lambda_i(D^2u)>0} \lambda_i(D^2u) + \Theta \sum_{\lambda_i(D^2u)<0} \lambda_i(D^2u)$$

and

$$\mathcal{M}^+_{\theta,\Theta}(D^2u) = \Theta \sum_{\lambda_i(D^2u)>0} \lambda_i(D^2u) + \theta \sum_{\lambda_i(D^2u)<0} \lambda_i(D^2u),$$

associated with the set of matrices

$$\mathcal{A} = \left\{ A \in S^n_+(\mathbb{R}) : \sqrt{\theta} \le \lambda_i(A) \le \sqrt{\Theta} \right\},$$

In fact, one can write

$$\mathcal{M}^-_{\theta,\Theta}(M) = \inf_{A\in\mathcal{A}} \mathrm{tr}(A^tMA) \qquad \text{and} \qquad \mathcal{M}^+_{\theta,\Theta}(M) = \sup_{A\in\mathcal{A}} \mathrm{tr}(A^tMA).$$

Examples

- The equation for the convex envelope (Oberman-Silvestre)

$$\lambda_1(D^2u) = \mathsf{min}\left\{\lambda : \lambda \text{ is an eigenvalue of } D^2u\right\},$$

that corresponds to the set of matrices

$$\mathcal{A} = \left\{A \in S^n_+(\mathbb{R}) : \lambda_1(A) = \cdots = \lambda_{n-1}(A) = 0 \text{ and } \lambda_n(A) = 1\right\}.$$

- Truncated Laplacians (Birindelli-Galise-Ishii)

$$\mathcal{P}^-_k(D^2u) = \sum_{i=1}^{k} \lambda_i(D^2u) \qquad \text{and} \qquad \mathcal{P}^+_k(D^2u) = \sum_{i=1}^{k} \lambda_{n+1-i}(D^2u),$$

for $k = 1, 2, \ldots, n-1$. Just take

$$\mathcal{A} = \left\{A : \lambda_1 = \cdots = \lambda_{n-k} = 0 \text{ and } \lambda_{n-k+1} = \cdots = \lambda_n = 1\right\}.$$

sup-inf operators

Our next step is to consider sup-inf operators, let $\mathbb{A}_x \subset \mathcal{P}(S^n(\mathbb{R}))$ be a non-empty subset for each $x \in \mathbb{R}^n$ and assume that

$$\bigcup \mathbb{A}_x = \left\{ A \in S^n(\mathbb{R}) \ : \ A \in \mathcal{A} \text{ for some } \mathcal{A} \in \mathbb{A}_x \right\} \quad \text{is bounded.}$$

Consider

$$F(x, D^2u(x)) = \sup_{\mathcal{A} \in \mathbb{A}_x} \inf_{A \in \mathcal{A}} \operatorname{trace}(A^t D^2 u(x) A).$$

theorem

A function u is a viscosity solution to

$$F(x, D^2u(x)) = \sup_{\mathcal{A}\in\mathbb{A}_x} \inf_{A\in\mathcal{A}} \operatorname{trace}(A^t D^2 u(x) A) = f(x)$$

if and only if

$$u(x) = \sup_{\mathcal{A}\in\mathbb{A}_x} \inf_{A\in\mathcal{A}} \fint_{B_\varepsilon(0)} u(x + Ay)\, dy - \frac{\varepsilon^2}{2(n+2)} f(x) + o(\varepsilon^2),$$

as $\varepsilon \to 0$.

Examples

- Isaacs operators

$$F(x, D^2u(x)) = \sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} \operatorname{trace}\left(A^t_{\alpha\beta} D^2u(x) A_{\alpha\beta}\right).$$

Remark: every uniformly elliptic operator can be written as an Isaacs operator.

- The k-th smallest eigenvalue of the Hessian,

$$\lambda_k\left(D^2u(x)\right) = \max_V \left\{ \min_{v\in V,\ |v|=1} \langle D^2u(x)v, v\rangle \ :\ \dim(V) = n-k+1 \right\}.$$

Take the set

$$\mathbb{A} = \left\{ \{A : \lambda_i(A) = 0 \text{ for } i \neq n, \lambda_n(A) = 1,\ \text{and } v_n \in V\} \right\},$$

$\dim(V) = n - k + 1.$

Operators involving unbounded sets of coefficients

Next, we consider operators that are obtained from unbounded sets of matrices,

$$F(D^2u) = \inf_{A \in \mathcal{A}} \operatorname{trace}(A^t D^2 u A).$$

We consider the set

$$\Gamma_{\mathcal{A}} = \left\{ M \in S^n(\mathbb{R}) : F(M) > -\infty \right\}$$

and assume that

$$F \text{ is continuous in } \Gamma_{\mathcal{A}}.$$

Operators involving unbounded sets of coefficients

We say that

$$u \in C^2(\Omega) \text{ is } \mathcal{A}\text{-admissible in } \Omega \text{ if}$$

$$D^2u(x) \in \Gamma_{\mathcal{A}} \text{ for every } x \in \Omega,$$

i.e.,

$$F(D^2u(x)) > -\infty$$

for every $x \in \Omega$.

This condition plays an analogous role to the convexity ($D^2u \geq 0$) for the Monge-Ampère equation.

theorem

Let $u \in C^2(\Omega)$ be an $\mathcal{A}$-admissible function. Then, for every $x \in \Omega$ we have

$$\inf_{\substack{A \in \mathcal{A} \\ A \leq (\varepsilon)^{-1/2} Id}} \fint_{B_\varepsilon(0)} u(x+Ay)\,dy - u(x) = \frac{\varepsilon^2}{2(n+2)} F(D^2 u(x)) + o(\varepsilon^2),$$

as $\varepsilon \to 0$.

As a consequence, u solves

$$F(D^2 u(x)) = f(x)$$

if and only if

$$u(x) = \inf_{\substack{A \in \mathcal{A} \\ A \leq (\varepsilon)^{-1/2} Id}} \fint_{B_\varepsilon(0)} u(x+Ay)\,dy - \frac{\varepsilon^2}{2(n+2)} f(x) + o(\varepsilon^2),$$

as $\varepsilon \to 0$.

Example. Monge-Ampere. Local version

It holds that

$$\det D^2u(x) = f(x),$$

if and only if

$$u(x) = \inf_{\substack{\det A=1 \\ A\leq(\varepsilon)^{-1/2}Id}} \left\{ \fint_{B_\varepsilon(0)} u(x+Ay)\,dy \right\} - \frac{\varepsilon^2\, n}{2(n+2)}\,(f(x))^{1/n} + o(\varepsilon^2)$$

as $\varepsilon \to 0$.

Example. Monge-Ampere. Nonlocal version

It holds that

$$\det D^2 u(x) = f(x),$$

if and only if

$$u(x) = \inf_{\substack{\det A=1 \\ x+AB_\varepsilon(0)\subset\Omega}} \left\{ \fint_{B_\varepsilon(0)} u(x+Ay)\,dy \right\} - \frac{\varepsilon^2 n}{2(n+2)} (f(x))^{1/n} + o(\varepsilon^2)$$

as $\varepsilon \to 0$.

Example. k−Hessians

k−Hessian operators, which are given by the elementary symmetric polynomials

$$\sigma_k(\lambda_1, \ldots, \lambda_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k}$$

evaluated in the eigenvalues of the Hessian, $\{\lambda_i(D^2u)\}_{1 \le i \le n}$.

For these operators to fit our framework we need to write them in the form

$$F_k(D^2u(x)) = k \left[\sigma_k\Big(\lambda_1(D^2u(x)), \ldots, \lambda_n(D^2u(x))\Big)\right]^{\frac{1}{k}},$$

Example. k−Hessians

In this case the result reads as:
Assume that $u \in C^2(\Omega)$ is k-convex, that is, $\sigma_j(\lambda(D^2u(x))) \geq 0$ for all $j = 1, \dots, k$, for every $x \in \Omega$. Then, for every $x \in \Omega$ we have

$$\inf_{\substack{A \in \mathcal{A}_k \\ A \leq (\varepsilon)^{-1/2} Id}} \fint_{B_\varepsilon(0)} u(x+Ay)\,dy - u(x) = \frac{\varepsilon^2}{2(n+2)} k(\sigma_k(D^2u(x)))^{\frac{1}{k}} + o(\varepsilon^2),$$

as $\varepsilon \to 0$, where

$$\mathcal{A}_k = \Big\{ A : \ \lambda_i^2(A) = \sigma_{k-1,i}(\gamma) \text{ with } \gamma = (\gamma_1, \dots, \gamma_n) \in \Gamma_k \text{ and } \sigma_k(\gamma) = 1 \Big\}$$

and $\sigma_{k-1,i}(\gamma_1, \dots, \gamma_n) = \sigma_{k-1}(\gamma_1, \dots, \gamma_{i-1}, 0, \gamma_{i+1}, \dots, \gamma_n)$.

Lower-order terms.

When the mean value formula involves averages over balls that are not centered at 0 but at $\varepsilon^2 v$ with $|v| = 1$, we obtain operators with first-order terms. For example, we have

$$\inf_{A\in\mathcal{A}_x} \fint_{B_\varepsilon(\varepsilon^2 v)} u(x+Ay)\,dy - u(x)$$
$$= \varepsilon^2 \inf_{A\in\mathcal{A}_x} \left\{ \frac{1}{2(n+2)} \mathrm{tr}(A^t D^2u(x)A) + \langle Du(x), Av\rangle \right\} + o(\varepsilon^2),$$

as $\varepsilon \to 0$.

We can also look for zero-order terms and consider

$$\inf_{A\in\mathcal{A}_x} (1-\alpha\varepsilon^2) \fint_{B_\varepsilon(0)} u(x+Ay)\,dy - u(x)$$
$$= \varepsilon^2 \left\{ \frac{1}{2(n+2)} \inf_{A\in\mathcal{A}_x} \mathrm{tr}(A^t D^2u(x)A) - \alpha u(x) \right\} + o(\varepsilon^2),$$

as $\varepsilon \to 0$.

Ingredients in the proofs.

- Mean values.

$$\fint_{B_\varepsilon(0)} c\, dy = c, \qquad c \in \mathbb{R},$$

$$\fint_{B_\varepsilon(0)} \langle v, y\rangle\, dy = 0, \qquad v \in \mathbb{R}^n,$$

$$\fint_{B_\varepsilon(0)} \langle My, y\rangle\, dy = \frac{\varepsilon^2}{n+2} \mathrm{trace}(M), \qquad M \in S^n.$$

- $M \mapsto F(x, M)$ is continuous in M

$$\inf_{A \in \mathcal{A}_x} \mathrm{trace}(A^t\,(M \pm \eta I)\,A) \to \inf_{A \in \mathcal{A}_x} \mathrm{trace}(A^t M A)$$

as $\eta \to 0$, for every $M \in S^n(\mathbb{R})$.

The heart of the matter. $u \in C^2$.

Given $x \in \Omega$, consider the paraboloid

$$P(z) = u(x) + \langle \nabla u(x), z - x \rangle + \frac{1}{2} \langle D^2 u(x)(z - x), (z - x) \rangle.$$

Since $u \in C^2$ we have

$$\begin{aligned} &\fint_{B_\varepsilon(0)} u(x + Ay)dy \approx \fint_{B_\varepsilon(0)} P(x + Ay)dy \\ &= \fint_{B_\varepsilon(0)} u(x)dy + \fint_{B_\varepsilon(0)} \langle \nabla u(x), Ay \rangle dy + \frac{1}{2} \fint_{B_\varepsilon(0)} \langle D^2 u(x) Ay, Ay \rangle dy \\ &= u(x) + \frac{1}{2} \frac{\varepsilon^2}{n + 2} \mathrm{trace}(A^t D^2 u(x) A). \end{aligned}$$

Then, we expect that,

$$\inf_{\mathcal{A}} \fint_{B_\varepsilon(0)} u(x + Ay)dy \approx u(x) + \frac{1}{2} \frac{\varepsilon^2}{n + 2} \inf_{\mathcal{A}} \mathrm{trace}(A^t D^2 u(x) A).$$

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Grazie !!!

Thanks !!!

Gracias !!!